Math 246A Lecture 20 Notes

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1 The Residue Theorem

1.1 The residue theorem

Let $f \in H(\{z : 0 < |z-a| < R\}$. Let $\gamma = \{z : |z-a| = r\}$ for $0 < r < R$. Then

$$\alpha = \frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

is independent of r. If the singularity is removable, then $\alpha = 0$.

Definition 1.1. The quantity α is called the **residue** of f at a.

Example 1.1. Let $f(z) = \sum j = 1^n c_j/(z-a_j) + g$, where g is holomorphic on $\{z : |z-a| < R\}$. Then $\alpha = c_1$.

Example 1.2. We can write $e^{1/z} = \sum_{n=0}^\infty \frac{1}{n!} z^{-n}$ for $0 < |z| < \infty$. Then $\alpha = 1$.

Theorem 1.1 (Residue theorem)**.** *Suppose that $f \in H(\Omega \setminus \{a_1, \dots, a_n\})$ with $a_j \in \Omega$. Let $\gamma \subseteq \Omega \setminus \{a_1, \dots, a_n\}$ be such that $\gamma \sim 0$ in Ω. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{j=1}^n \operatorname{Res}(f, a_j) n(\gamma, a_j)$$

Proof. This follows from the general form of Cauchy's theorem. $\square$

1.2 Solving real integrals using contour integration

We can use the residue theorem to solve real integrals.

Example 1.3. Use the change of variables $z = e^{i\theta}$.

$$\int_0^{2\pi} \frac{1}{3+\sin(\theta)}\, d\theta = \oint_{|z|=1} \frac{1}{3 + (z-1/z)/2i} \frac{1}{iz}\, dz = \oint \frac{2}{z^2 + 6iz + 1}\, dz$$

If we factor the bottom and apply the residue theorem, we get $2\pi/\sqrt{8}$.

Remark 1.1. A good way to check you answer in a case like the above example is to check whether the answer is real. It should be because the integrand was real.

Example 1.4.

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx = \lim_{R\to\infty} \operatorname{Re} \int_{-R}^{R} \frac{e^{iz}}{z(1+z^2)}\,dz$$

This is the integral over real-line part of a upper semicircle contour. Call this γ_1, and let γ_2 be the circle part of the semicircle contour. Using the residue theorem, the real part of the integral over $\gamma_1+\gamma_2$ is π/e, and we just need to show that the integral over γ_2 goes to 0 as $R\to\infty$.

Example 1.5.

$$\int_{-\infty}^{\infty} \frac{x\sin(\lambda x)}{1+x^2}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{ze^{i\lambda z}}{1+z^2}\,dz = \lim_{A,B\to\infty} \operatorname{Im} \int_{-A}^{B} \frac{ze^{i\lambda z}}{1+z^2}\,dz$$

View this as the integral over γ_1, the real-line part of a rectangle $\gamma_1+\gamma_2+\gamma_3+\gamma_4$ with vertices $-A, B, B+i\min(A,B), -A+i\min(A,B)$ that extends into the upper half plane.

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} = 2\pi i(ie^{-\lambda}/2i) = \pi e^{-\lambda} i$$

$$\left|\int_{\gamma_3}\right| \leq \frac{(A+B)(2(A+B))}{(A+B)^2-1} e^{-\lambda(A+B)} \to 0$$

$$\left|\int_{\gamma_2}\right| \leq \int_0^B \frac{B}{B^2-1} e^{-\lambda y}\,dy \to 0$$

$$\left|\int_{\gamma_4}\right| \leq \int_0^B \frac{A+B}{A^2-1} e^{-\lambda y}\,dy,$$

which should go to zero. We will finish this next time.